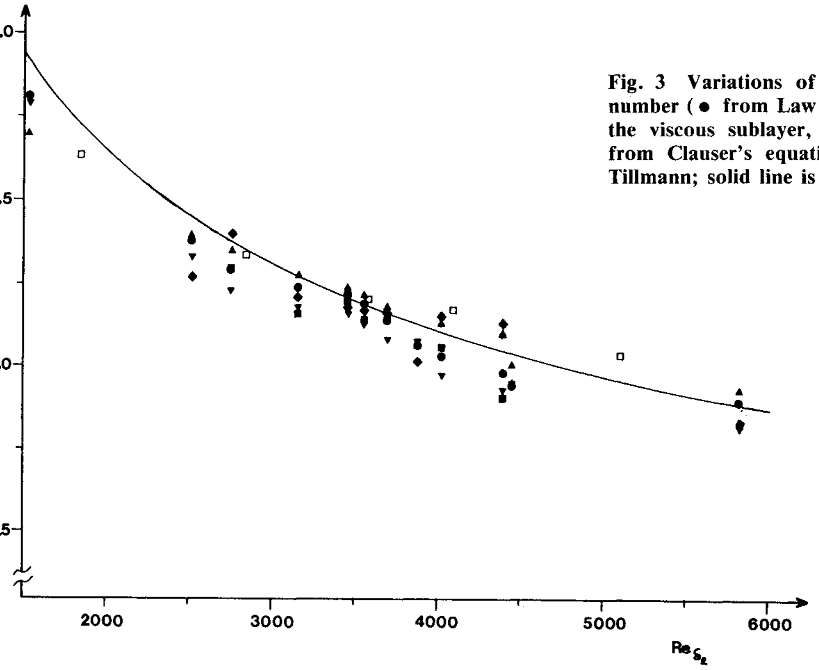


Fig. 3 Variations of the skin-friction coefficient with Reynolds number (● from Law of the wall, ■ from mean velocity profile in the viscous sublayer, ◆ from momentum/integral equations, ▼ from Clauser's equation, ▲ from the relation of Ludwieg and Tillmann; solid line is Coles' distribution).

gradient of the momentum-loss thickness with reasonable accuracy. Both Clauser's relation and Ludwieg and Tillmann's relation for $c_f$ are simple to use because no gradients of parameters entering the equations are required. They do require integral parameters be calculated and will, therefore, be sensitive to any flow anomalies in the outer part of the boundary layer. The overall conclusion from the present study is that Spalding's law of the wall (when applied to the wall region only) is to be preferred for the determination of the local wall shear stress.

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# Wind-Tunnel Wall Corrections on a Two-Dimensional Plate by Conformal Mapping

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## I. Introduction

THE problem of wall interference is of practical interest, because in aerodynamics it is not always possible to test a model in unconstrained freestream flow. This problem is dealt with in Ref. 1. One approximate formula given there treats the case in which the profile is a plate in a two-dimensional, steady, and irrotational ideal flow (i.e., inviscid and incompressible). The general drawback of such formulas is their inaccuracy when the airfoil has relatively large chord $c$.

This Note introduces a conformal mapping method for computing this ideal flow and the resulting lift exactly. In our method, the domain between the profile and the tunnel walls is mapped conformally onto an annulus using a Schwarz-Christoffel map for doubly connected regions (see Ref. 2, Sec. 17.5). In order to compute this map numerically, we have to

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solve a parameter problem, which is done by analogy with the simply connected case.[3-5] To the best of our knowledge, Schwarz-Christoffel maps for general, doubly connected polygonal geometries have not been computed before.

Once transplanted to the annulus, the flow problem can be solved directly. The solution is not unique, but it becomes unique if we determine the circulation by the Kutta-Joukowski condition. By transplanting back, we then obtain the solution of the original problem.

## II. Conformal Mapping and Two-Dimensional Ideal Flow

Let $x,y$ be Cartesian coordinates in the physical plane containing the horizontal channel and the profile, let $z=x+iy$ be the corresponding complex variable, and let $q(x,y) = [q_1(x,y), q_2(x,y)]$ be the velocity vector of the flow at $(x,y)$. Our objective is to solve the following flow problem: given a polygonally shaped profile, calculate the velocity field $q(z)$ for the ideal flow through the channel over this profile and the resulting lift. See Fig. 1.

If we assume the velocity far downstream of the airfoil to be $V$, so that

$$\lim_{Re(z)\to\infty} q(z) = q_\infty = (V,0), \quad V>0$$

and choose the circulation around the profile (in an appropriate way), there is exactly one solution to our problem and there is an analytic function $F(z)$, so that $\overline{F'(z)} = q(z) = q_1 + iq_2$, $F(z)$ is called the complex potential.

We use now the following fact. If a conformal mapping $f(z)=w$ exists that maps the region D between the profile and the tunnel walls onto a model region A (an annulus) and if we have a potential $G(w)$ that solves the transplanted flow problem in the annulus, then the potential we are looking for is

$$F(z) = G(f(z)) \tag{1}$$

and the complex velocity $q(z)=q_1+iq_2$ is given by $\overline{q(z)} = \overline{G'(w)f'(z)}$, where $w=f(z)$. (For theoretical background see Ref. 6, pp. 335 and 356.) We next discuss how to construct this mapping $f$.

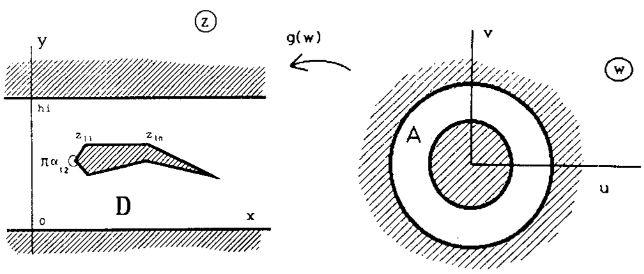


**Fig. 1 Doubly connected polygonal region and its conformally equivalent annulus.**

## III. Conformal Mapping of Doubly Connected Polygonal Regions

The Schwarz-Christoffel formula, which gives an almost explicit solution for the mapping of the unit disk onto an arbitrary simply connected region bounded by a polygon, has been known for more than a century. It is less well known that there exists an analogous formula for the map of an annulus A onto a *doubly connected* polygonal region D.

We assume that D is a bounded doubly connected region whose outer and inner boundary curves, $\Gamma_0$ and $\Gamma_1$, are closed polygonal lines. Let the consecutive vertices of $\Gamma_0$ be $z_{01}, z_{02}, \ldots, z_{0m}$ and let the vertices of $\Gamma_1$ be $z_{11}, z_{12}, \ldots, z_{1n}$. The angle at $z_{jk}$, measured from the interior of D, is denoted by $\alpha_{jk}\pi$, $0<\alpha_{jk}\le 2$.

*Theorem*: The function $z=g(w)$ that maps the conformally equivalent annulus A, $0<\mu<|w|<1$, onto the doubly connected polygonal region D is of the form

$$g(w) = g(w_c) + C\int_{w_c}^{w} \prod_{k=1}^{m} \left[\theta_0\left(\frac{w^*}{w_{0k}}\right)\right]^{\alpha_{0k}-1} \times \prod_{k=1}^{n} \left[\theta_1\left(\frac{w^*}{w_{1k}}\right)\right]^{\alpha_{1k}-1} \mathrm{d}w^* \tag{2}$$

where $C$ is a complex constant, $w_{0k}$ and $\mu w_{1k}$ the preimages of $z_{0k}$ and $z_{1k}$, respectively, and the theta functions $\theta_j$ are defined by

$$\theta_0(w) = (1-w)\prod_{v=2,4,6\ldots}^{\infty} (1-\mu^v w)(1-\mu^v w^{-1})$$

$$\theta_1(w) = \prod_{d=1,3,5\ldots}^{\infty} (1-\mu^d w)(1-\mu^d w^{-1})$$

*Proof*: See Ref. 2, Sec. 17.5.

We mention that this theorem also holds if one or several vertices $z_{0k}$ are at infinity. The constants $C$, $\mu$, $w_{0k}$ $(k=1,\ldots,m)$ and $w_{1k}$ $(k=1,\ldots,n)$ in Eq. (2) are a priori unknown. Because it is possible to map the annulus A onto itself by a rotation, one of the $w_{0k}$ can be chosen arbitrarily on the unit circle. The remaining $m+n+2$ real constants $\mathrm{Re}(C)$, $\mathrm{Im}(C)$, $\mu$, $\arg(w_{01}),\ldots\arg(w_{0,m-1})$, $\arg(w_{11})$, $\ldots\arg(w_{1n})$ are called the *accessory parameters* of the problem. In most cases, they can be determined only by numerical computation. They satisfy the equations

$$z_{01} - z_{0m} = C\int_{w_{0m}}^{w_{01}} \ldots \mathrm{d}w, \quad \left| z_{0,k+1} - z_{0k} \right| = \left| C\int_{w_{0k}}^{w_{0,k+1}} \ldots \mathrm{d}w \right| \quad k=1,2,\ldots m-1 \tag{3}$$

$$z_{11} - z_{01} = C\int_{w_{01}}^{w_{11}} \ldots \mathrm{d}w \quad \text{(2 real equations)} \tag{4}$$

$$z_{11} - z_{1n} = C\int_{w_{1n}}^{w_{11}} \ldots \mathrm{d}w, \quad |z_{1,k+1} - z_{1k}| = \left| C\int_{w_{1k}}^{w_{1,k+1}} \ldots \mathrm{d}w \right| \quad k=1,2,\ldots n-3 \tag{5}$$

where $\ldots\mathrm{d}w$ is defined by

$$\prod_{k=1}^{m} \left[\theta_0\left(\frac{w}{w_{0k}}\right)\right]^{\alpha_{0k}-1} \prod_{k=1}^{n} \left[\theta_1\left(\frac{w}{w_{1k}}\right)\right]^{\alpha_{1k}-1} \mathrm{d}w$$

Equations (3) ensure that $C\int_\Gamma \ldots \mathrm{d}w = 0$, when $\Gamma$ is the circle $\rho e^{i\tau}$, $0\le\tau\le 2\pi$ and $\rho\epsilon(\mu,1)$. This implies that $g(w_c) + \int_{w_c}^{w} \ldots \mathrm{d}w$ represents a single-valued function. Equations (3-5) ensure that the geometry of the doubly connected polygonal region D will come out right. Thus, we have a nonlinear system of $n+m+2$ equations for the $n+m+2$ unknowns: $\phi_{01}$, $\phi_{02}, \ldots \phi_{0,m-1}$, $\phi_{11}$, $\phi_{12}, \ldots \phi_{1n}$, $\mu$, $\mathrm{Re}(C)$, $\mathrm{Im}(C)$, where $\phi_{jk} = \arg(w_{jk})$. To most easily enforce the constraints

$$0<\phi_{01}<\phi_{02}\ldots<\phi_{0,m=1}<2\pi,$$

$$\phi_{1n}<\phi_{11}<\phi_{12}\ldots<\phi_{1,n-1}<\phi_{1n}+2\pi, \quad 0<\mu<1$$

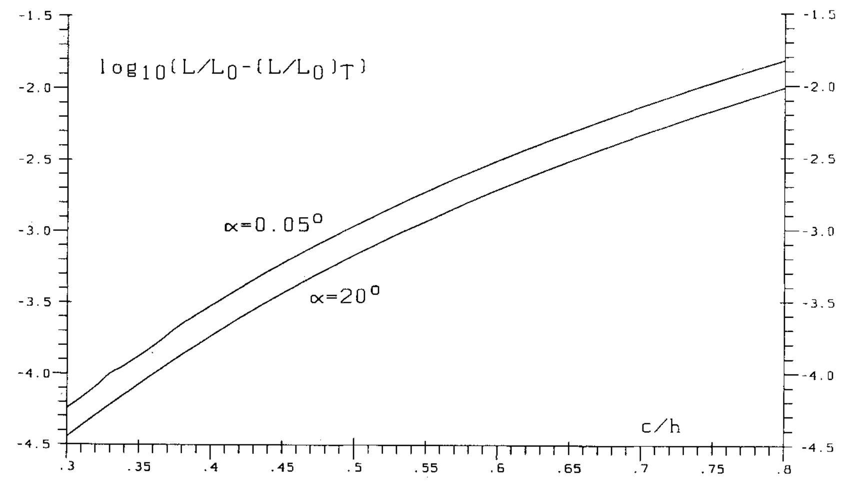


Fig. 2 Error of Tomotika's formula.

we can transform these unknowns according to

$$y_1=\frac{1}{0.98-\mu}-\frac{1}{\mu-0.02},\quad y_2=\mathrm{Re}(C),\quad y_3=\mathrm{Im}(C)$$

$$y_{3+k}=\ell n\left(\frac{\phi_{0k}-\phi_{0,k-1}}{\phi_{0,k+1}-\phi_{0k}}\right),$$

$$k=1,\ldots m-1,\quad \phi_{00}=0,\quad \phi_{0m}=2\pi$$

$$y_{m+3}=\frac{1}{4-\phi_{1n}}-\frac{1}{\phi_{1n}+4} \tag{6}$$

and the remaining unknowns $\phi_{11}$, $\phi_{12}$,...$\phi_{1,n-1}$ are also transformed by Eq. (6), which was proposed by Trefethen[3,7] in the simply connected case.

To solve the resulting unconstrained nonlinear system of equations we used the MINPACK program HYBRD, which is based on the hybrid methods developed by Powell.[8,9]

Whenever we want to evaluate the integral in Eq. (2), we have to compute theta functions. Since $\theta_0(w)=\theta_1(\mu^{-1}w)$ and

$$\theta_1(w)=1\Big/\prod_{k=1}^{\infty}(1-\mu^{2k})\sum_{k=-\infty}^{\infty}\mu^{k^2}(-w)^k$$

it suffices to compute

$$\sigma_1(w)=1+\sum_{k=1}^{\infty}\mu^{k^2}(w^k+w^{-k})$$

which can be done efficiently (see also Ref. 2, Sec. 17.5).

If at least one end point of the path of integration is the preimage of the vertex $z_{jk}$, then the integrand in Eq. (2) has, near $g^{-1}(z_{jk})$, the form $h(w)[w-g^{-1}(z_{jk})]^{\alpha_{jk}-1}$ with some analytic function $h(w)$ and we can use the appropriate Gauss-Jacobi quadrature formula to take this singularity into account.

## IV. Computation of Flow and Lift

For a given configuration, we proceed as follows:

1) Determine the mapping $g(w)$ of the annulus A onto the region D between the profile and the walls, as described in Sec. III. The inverse $g^{-1}$ of $z_0\epsilon$ D is then the zero of $g(w)-z_0=0$ and can be found by Newton's iteration whenever a good approximation to $w=g^{-1}(z_0)$ is already known.

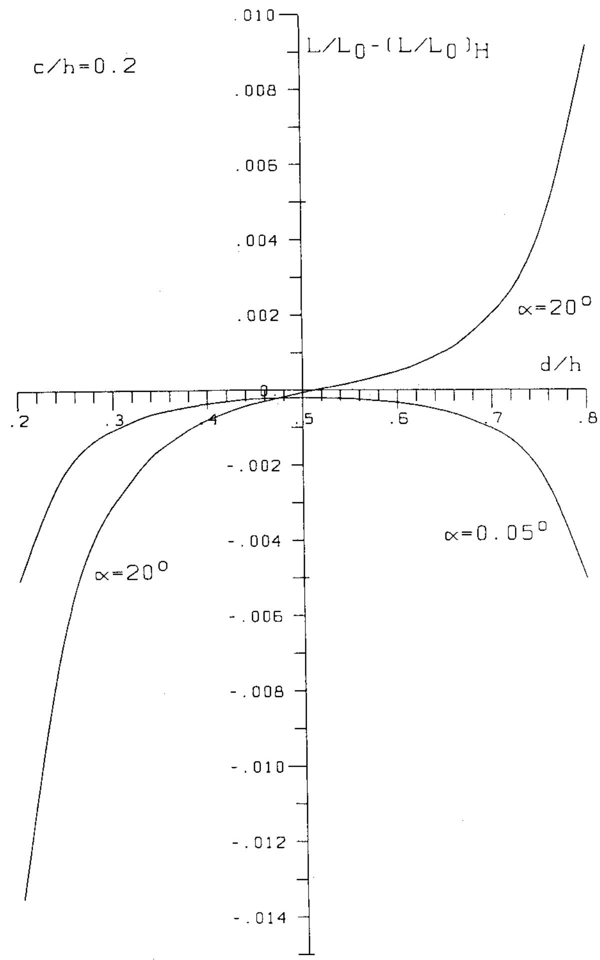


Fig. 3 Error of Havelock's formula.

2) Derive the potential $G(w)$ in the model region A that solves the transplanted flow problem. It can be shown[10] that

$$G(w)=\frac{Vh}{\pi}\left\{\ell n\left[\theta_0\left(\frac{w}{w_{01}}\right)\right]-\ell n\left[\theta_0\left(\frac{w}{w_{02}}\right)\right]\right\}-\gamma i\ell n(w) \tag{7}$$

where $\gamma\epsilon R$ and $w_{01}$ and $w_{02}$ are the images of $-\infty$ and $\infty$ in the annulus A, respectively. Because of Kutta-Joukowski condition, the trailing edge $z^*$ of the profile has to coincide with a stagnation point, i.e., one has to choose $\gamma$ so that $G'(w^*)=0$, for $w^*=g^{-1}(z^*)$.

3) Transplant back to get the solution $F(z)$ of the original flow problem. From Eq. (1), there follows $F(z)=G[g^{-1}(z)]$.

4) Compute force $L$ exerted on the profile according to Blasius' theorem,

$$L=-\frac{i\rho}{2}\int_\Gamma \overline{|q|^2 dz}=-\frac{i\rho}{2}\int_\Gamma \overline{[F'(z)]^2 dz} \tag{8}$$

where $\Gamma$ is any positively oriented curve in D encircling the profile and $\rho$ the density (see Ref. 6, p. 366).

## V. Numerical Results

In Ref. 1, two approximate formulas are given for the lift $L$ of a flat-plate airfoil inclined at an arbitrary angle $\alpha$ to an incompressible stream between parallel walls. $L$ will depend on the height $h$ of the tunnel, the chord $c$, and the angle of incidence $\alpha$ of the airfoil and on the distance $d$ of its midpoint from the floor of the tunnel.

For the midchord of the plate in the centerline of the tunnel, i.e., $d=0.5h$, Tomotika (see Ref. 1, p. 34) derives

$$\frac{L}{L_0}=1+\frac{\pi^2}{24}(1+\sin^2\alpha)\left(\frac{c}{h}\right)^2 - \frac{\pi^4}{7680}(11-53\sin^2\alpha-22\sin^4\alpha)\left(\frac{c}{h}\right)^4+\mathcal{O}\left[\left(\frac{c}{h}\right)^6\right] \tag{9}$$

where $L_0=i\rho\,|q_\infty|^2 c\pi\sin\alpha$ is the freestream lift.

We compare now the tunnel to free-air ratios $(L/L_0)_T$ estimated by Eq. (9) with our exact values $L/L_0$ computed by Eq. (8).

In Fig. 2, the values $\log_{10}[L/L_0-(L/L_0)_T]$ for $\alpha=0.05$ and 20 deg are plotted against $c/h$, which makes sense, as $(L/L_0)_T$ was always smaller than $(L/L_0)$. When $\alpha=0.05$ deg and $c<0.33h$, the error of Eq. (9) is $<0.0001$, but it exceeds 0.01 when $c=0.75h$. For the corresponding values of $\alpha=20$ deg, the error is $<0.0001$ for $c<0.36h$, but it exceeds 0.001 when $c=0.53h$.

Havelock (Ref. 1, p. 34) gives a formula for $L/L_0$ for a plate whose midpoint is at an arbitrary distance $d$ from the floor of the tunnel,

$$\frac{L}{L_0}=1-\frac{\pi}{2}\cot\left(\frac{\pi d}{h}\right)\sin\alpha\left(\frac{c}{h}\right)+\frac{\pi^2}{16}\left\{\left[\frac{2}{3}+\cot^2\left(\frac{\pi d}{h}\right)\right] + \left[\frac{2}{3}+3\cot^2\left(\frac{\pi d}{h}\right)\right]\sin^2\alpha\right\}\left(\frac{c}{h}\right)^2+\mathcal{O}\left[\left(\frac{c}{h}\right)^3\right] \tag{10}$$

Figure 3 shows the variation of $L/L_0-(L/L_0)_H$ with the ratio $d/h$ where $(L/L_0)_H$ is computed by Eq. (10) for selected values of $\alpha$ and $c=0.2h$. Even for relatively small chord $c$ (i.e., $c=0.2h$), Havelock's formula is not very accurate for $\alpha=20$ deg and $d=0.8h$: Eq. (10) yields $L/L_0=1.22941$, whereas we obtained $L/L_0=1.23857$.

## Acknowledgments

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# Approximate Analysis of Deflections and Frequencies of Short Beams

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## Introduction

Many papers have been published on the analysis of stresses, deflections, and frequencies of short beams. For instance, considering the warping of a section, Murty[1,2] and Levinson[3,4] introduced the equations of motion and Rehfield and Murthy,[5] assuming the stress function as $\Sigma_m\Sigma_n a_{mn}x^m y^n$, determined the stress distribution of a beam subjected to a uniformly distributed load. Several questionable points in their methods and results were discussed in detail in Ref. 6.

To remove these defects, assuming the axial normal stress $\sigma_x$ as $\Sigma_n y^n u_n(x)$, the author[6] obtained the shearing stress $\tau$ and normal stress $\sigma_y$ in the transverse direction by the equilibrium conditions and determined $u_n$, using the minimum complementary energy principle. However, dynamical cases cannot be analyzed by this method.

The following method will be applied in this Note: The axial displacement $u$ is assumed to be in the form of $yu_1+Yu_2$, where $Y$ is a specified odd function with respect to $y$. In the first step, $u_1$ and $u_2$ are determined by solving the fundamental equations obtained from the equilibrium conditions. In the second step, on the assumption of $Y=\Sigma_m b_m y^m$, the coefficients $b_m$ are determined as to minimize the total energy $V$ for static loads and to satisfy the condition $V_s=V_p$ for dynamic loads, where $V_s$ and $V_p$ are the strain and potential energies, respectively. The latter condition is not satisfied in Levinson's method.[3]

## Determination of Deflections and Stresses Under Statical Loads

In order to simplify the calculations, the case will be considered where a beam with a rectangular cross section is sub-

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